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## High-precision estimate of $g_4$ in the 2D Ising model

Michele Caselle<sup>a</sup>, Martin Hasenbusch<sup>b</sup>, Andrea Pelissetto<sup>c</sup> and Ettore Vicari<sup>d</sup>

<sup>a</sup> *Dipartimento di Fisica Teorica dell'Università di Torino and I.N.F.N., I-10125 Torino, Italy*

<sup>b</sup> *Humboldt Universität zu Berlin, Institut für Physik, Invalidenstr. 110, D-10115 Berlin, Germany*

<sup>c</sup> *Dipartimento di Fisica dell'Università di Roma I and I.N.F.N., I-00185 Roma, Italy*

<sup>d</sup> *Dipartimento di Fisica dell'Università di Pisa and I.N.F.N., I-56127 Pisa, Italy*

### Abstract

We compute the renormalized four-point coupling in the 2d Ising model using transfer-matrix techniques. We greatly reduce the systematic uncertainties which usually affect this type of calculations by using the exact knowledge of several terms in the scaling function of the free energy. Our final result is  $g_4 = 14.69735(3)$ .

# 1 Introduction

An important information on the physical properties of a quantum field theory is given by the renormalized four-point coupling, which is defined in terms of the zero momentum projection of the truncated 4-point correlator. At the same time, if one is interested in the lattice discretization of the theory, this renormalized coupling represents one of the most interesting universal amplitude ratios, being related to the fourth derivative of the free energy.

Recently, in [1], a new interesting approach has been proposed to evaluate this quantity in the case of integrable QFT's. The idea is that for these theories one has direct access to the so called form factors from which the renormalized coupling can be computed. In [1] the method was tested in the case of the 2d Ising model. The authors found the remarkably precise estimate

$$g_4^* = 14.6975(1) \tag{1}$$

(see below for the precise definition of  $g_4^*$ ).

The aim of this paper is to test this result by using a completely different method. By combining transfer-matrix methods and the exact knowledge of several terms in the scaling function of the free energy of the model we are able to obtain a precision similar to that of [1]. Our result is

$$g_4^* = 14.69735(3), \tag{2}$$

which is in substantial agreement with the estimate of [1]. Given the subtlety of the calculations involved in both approaches, our result represents a highly non-trivial test of both methods. In performing our analysis we employ the same techniques which were used in [2] in the study of the 2d Ising model in a magnetic field.

This paper is organized as follows: We begin in sect.2 by collecting some definitions and elementary results which will be useful in the following. Sect.3 is devoted to a discussion of the transfer-matrix results (and of the techniques that we use to improve the performance of the method). In sect.4 we obtain the first 4 terms of the scaling function for the fourth derivative of the free energy, which enters in the estimate of  $g_4$  and finally, in sect.5, we discuss the fitting procedure that we used to extract the continuum-limit value from the data. To help the reader to follow our analysis, we have listed in tab.2 the output of our transfer-matrix analysis.

## 2 General setting

We are interested in the 2d Ising model defined by the partition function

$$Z = \sum_{\sigma_i = \pm 1} e^{\beta \sum_{\langle n, m \rangle} \sigma_n \sigma_m + h \sum_n \sigma_n}, \quad (3)$$

where the field variable  $\sigma_n$  takes the values  $\{\pm 1\}$ ;  $n \equiv (n_0, n_1)$  labels the sites of a square lattice of size  $L_0$  and  $L_1$  in the two directions and  $\langle n, m \rangle$  denotes nearest neighbour sites on the lattice. In our calculations with the transfer-matrix method we shall treat asymmetrically the two directions. We shall denote  $n_0$  as the “time” coordinate and  $n_1$  as the “space” one. The number of sites of the lattice will be denoted by  $N \equiv L_0 L_1$ . The critical value of  $\beta$  is

$$\beta = \beta_c = \frac{1}{2} \log(\sqrt{2} + 1) = 0.4406868\dots$$

In the following we shall be interested in the high-temperature phase of the model in which the  $\mathbf{Z}_2$  symmetry is unbroken, i.e. in the region  $\beta < \beta_c$ . It is useful to introduce the reduced temperature  $t$  defined as:

$$t \equiv \frac{\beta_c - \beta}{\beta_c}. \quad (4)$$

As usual, we introduce the free-energy density  $F(t, h)$  and the magnetization per site  $M(t, h)$  defined as

$$F(t, h) \equiv \frac{1}{N} \log(Z(t, h)), \quad M(t, h) \equiv \frac{\partial F(t, h)}{\partial h}. \quad (5)$$

The standard definition of the four-point zero-momentum renormalized coupling constant  $g_4$  is

$$g_4(t) = -\frac{F^{(4)}}{\chi^2 \xi_{2\text{nd}}^2}, \quad (6)$$

where  $\chi$  and  $F^{(4)}$  are the second- and fourth-order derivatives of the free-energy density  $F(h, t)$  at  $h = 0$ :

$$\chi(t) = \left. \frac{\partial^2 F(t, h)}{(\partial h)^2} \right|_{h=0}, \quad F^{(4)}(t) = \left. \frac{\partial^4 F(t, h)}{(\partial h)^4} \right|_{h=0} \quad (7)$$

and  $\xi_{2\text{nd}}$  denote the second moment correlation length. The second moment correlation length is defined by

$$\xi_{2\text{nd}}^2 = \frac{\mu_2}{2d\mu_0} , \quad (8)$$

where  $d$  is the dimension (here  $d = 2$ ) and

$$\mu_i = \lim_{L_1 \rightarrow \infty} \lim_{L_0 \rightarrow \infty} \frac{1}{N} \sum_{m,n} (m-n)^i \langle \sigma_m \sigma_n \rangle_c . \quad (9)$$

The connected part of the correlation function is given by

$$\langle \sigma_m \sigma_n \rangle_c = \langle \sigma_m \sigma_n \rangle - \langle \sigma_m \rangle \langle \sigma_n \rangle . \quad (10)$$

In particular, we are interested in the continuum-limit value  $g_4^*$  defined as

$$g_4^* = \lim_{t \rightarrow 0} g_4(t) . \quad (11)$$

For  $t \rightarrow 0$  we have

$$\xi_{2\text{nd}}(t) \simeq A_{\xi,2\text{nd}} t^{-1}, \quad \chi(t) \simeq A_\chi t^{-7/4}, \quad F^{(4)}(t) \simeq A_{F^{(4)}} t^{-11/2}, \quad (12)$$

from which it follows

$$g_4^* = -\frac{A_{F^{(4)}}}{A_\chi^2 A_{\xi,2\text{nd}}^2} . \quad (13)$$

The amplitude  $A_\chi$  is known exactly (see e.g. ref. [3]):  $A_\chi = 0.9625817322\dots$ . The amplitude  $A_{\xi,2\text{nd}}$  can also be computed exactly. Indeed, consider the exponential correlation length  $\xi$  (inverse mass gap). For  $t \rightarrow 0$ , it behaves as [3]  $A_\xi t^{-1}$ , with  $A_\xi = 1/(4\beta_c) = 0.56729632855\dots$ . Using then [4]  $A_\xi/A_{\xi,2\text{nd}} = 1.000402074\dots$ , we obtain finally  $A_{\xi,2\text{nd}} = 0.5670683251\dots$ .

Our goal in the remaining part of this paper is to give a numerical estimate of  $A_{F^{(4)}}$ .

### 3 Transfer-matrix results

We may have direct numerical access to  $F^{(4)}$  by looking at the  $h$  dependence of the magnetization at fixed  $t$ . Expand as follows:

$$h = b_1 M + b_3 M^3 \dots \quad (14)$$

we immediately see that

$$b_1 = 1/\chi, \quad b_3 = -\frac{F^{(4)}}{6\chi^4}, \quad (15)$$

so that

$$F^{(4)} = -6b_3/b_1^4. \quad (16)$$

### 3.1 The transfer-matrix technique

As a first step we computed the magnetization  $M$  of a system with  $L_0 = \infty$  and finite  $L_1$ . The magnetization of this system is given by

$$M = v_0^T \tilde{M} v_0, \quad (17)$$

where  $v_0$  is the eigenvector of the transfer matrix with the largest eigenvalue and  $\tilde{M}$  is a diagonal matrix with  $\tilde{M}_{ii}$  being equal to the magnetization of the time-slice configuration  $i$ . For a detailed discussion of the transfer-matrix method see e.g. ref. [5].

We computed  $v_0$  using the most trivial iterative method,

$$v_0^{n+1} = \frac{T v_0^n}{|T v_0^n|}, \quad (18)$$

starting from a vector with all entries being equal.

An important ingredient in the calculation is the fact that the transfer matrix can be written as the product of sparse matrices (see e.g. ref. [6]). This allows us to reach  $L_1 = 24$  on a workstation. The major limitation is the memory requirement. We have to store two vectors of size  $2^{L_1}$ . Since we performed our calculation in double precision, this means that 268 MB are needed. Slightly larger  $L_1$  could be reached by using a super-computer with larger memory space.

For the parameters  $\beta$  and  $h$  that we studied,  $n \leq 200$  was sufficient to converge within double-precision accuracy.

### 3.2 The equation of state

In order to obtain high-precision estimates of  $F^{(4)}$  it turns out to be important to consider the external field  $h$  as a function of the magnetization rather than

the opposite. The advantage of the series (14) is that the coefficients — at least those we can compute — are all positive, and therefore, truncation errors are less severe than in the case of  $m(h)$ .

There is no sharp optimum in the truncation order. After a few numerical experiments we decided to keep in eq. (14) the terms up to  $b_{15} M^{15}$ :

$$h(M) = b_1 M + b_3 M^3 + \dots + b_{15} M^{15} \quad . \quad (19)$$

In order to compute the coefficients  $b_1, b_3, \dots, b_{15}$  we solved the system of linear equations that results from inserting 8 numerically calculated values of the magnetization  $M(h_1), M(h_2), \dots, M(h_8)$  into the truncated equation of state (19). Here we have chosen  $h_j = j h_1$ .

The errors introduced by the truncation of the series decrease as  $h_1$  decreases, while the errors from numerical rounding increase as  $h_1$  decreases. Therefore, we varied  $h_1$  to find the optimal choice. For a given value of  $\beta$  we performed this search only for one lattice size  $L_1$ . (Typically  $L_1 = 18$ ). From the variation of the result with  $h_1$  we can infer the precision of our estimates of  $b_i$ . For example, for  $\beta = 0.37$ , we get  $b_1$  with 14 significant digits and  $b_3$  with 12 significant digits.

### 3.3 Extrapolation to the thermodynamic limit

From the transfer matrix formalism it follows that for periodic boundary conditions and  $\beta \neq \beta_c$ , the free energy density approaches its thermodynamic limit value exponentially in  $L_1$ . Hence, also derivatives of the free energy density with respect to  $h$  and linear combinations of them should converge exponentially in  $L_1$  to their thermodynamic limit value. Therefore, in the simplest case, one would extrapolate with an Ansatz

$$b(L_1) = b(\infty) + c \exp(-xL_1) \quad , \quad (20)$$

where  $b(L_1)$  is the quantity at the given lattice size  $L_1$  and  $b(\infty)$  the thermodynamic limit of the quantity. In order to obtain numerical estimates for  $b(\infty)$ ,  $c$  and  $x$  we have inserted the numerical result of  $b$  for the three lattice sizes  $L_1$ ,  $L_1 - 1$  and  $L_1 - 2$  into eq. 20. It turns out that using this simple extrapolation, still a dependence of the result for  $b(\infty)$  on  $L_1$  is visible. This indicates that, with our numerical precision, subleading exponential corrections have to be taken into account. For this purpose we have iterated the extrapolation discussed above.

Table 1: *Extrapolation of  $b_3$  to the thermodynamic limit for  $\beta = 0.37$ . Iterative procedure. The numbers in the top row give the extrapolation level. For the discussion see the text.*

$L_1$	0	1	2	3	4
13	0.0459057204193				
14	0.0463456447150				
15	0.0467262982921	0.049170965			
16	0.0470483839889	0.048819648			
17	0.0473162239288	0.048638691	0.048446477		
18	0.0475358880370	0.048537476	0.048409004		
19	0.0477140164478	0.048477931	0.048392846	0.048380598	
20	0.0478571162217	0.048441711	0.048385463	0.048379249	
21	0.0479711755247	0.048419155	0.048381922	0.048378658	0.048378196
22	0.0480614838402	0.048404863	0.048380148	0.048378366	0.048378082
23	0.0481325804063	0.048395686	0.048379222	0.048378214	0.048378046
24	0.0481882778282	0.048389731	0.048378721	0.048378128	0.048378019

The iteration starts with  $b^{(0)}(L_1)$  which are the quantities  $b$  that have been computed by the transfer matrix for the lattice size  $L_1$ . A step of the iteration is given by solving the system of equations

$$\begin{aligned} b^{(i)}(L_1 - 2) &= c \exp(-x(L_1 - 2)) + b^{(i+1)}(L_1) \\ b^{(i)}(L_1 - 1) &= c \exp(-x(L_1 - 1)) + b^{(i+1)}(L_1) \\ b^{(i)}(L_1) &= c \exp(-xL_1) + b^{(i+1)}(L_1) . \end{aligned} \quad (21)$$

with respect to  $b^{(i+1)}(L_1)$ ,  $c$  and  $x$ . In table 1 we give as an example the extrapolation of  $b_3$  at  $\beta = 0.37$ . In the second column we give the results obtained for the given  $L_1$ . The stability of the extrapolation with varying  $L_1$  increases up to the fourth iteration. Further iterations become numerically unstable.

As final result we took  $b_3 = 0.04837802(3)$  from the 4<sup>th</sup> iteration. The error was estimated from the variation of the results with  $L_1$ . As a consistency check, we also extracted the thermodynamic limit by fitting with multi-exponential Ansätze. We found consistent results. The relative accuracy of  $b_1$  in the thermodynamic limit was in general better than that of  $b_3$ .

In the second column of table 2 we give our final results for  $-F^{(4)}t^{11/2}$  at the  $\beta$  values that we have studied. For a discussion of the following columns see section 5.

## 4 Scaling function for $F^{(4)}$

In this Section we shall study the asymptotic behavior of  $F^{(4)}(t)$  for  $t \rightarrow 0$  following Ref. [8]. With respect to [8], we have added the contributions due to the irrelevant operators. Here, we shall use the knowledge of the operator content of the theory at the critical point which can be obtained by using the methods of 2d conformal field theories.

General renormalization-group (RG) arguments indicate that the free energy of the model can be written as

$$\begin{aligned} F(t, h) &= F_b(t, h) + |u_t|^{d/y_t} f_{\text{sing}} \left( \frac{u_h}{|u_t|^{y_h/y_t}}, \left\{ \frac{u_j}{|u_t|^{y_j/y_t}} \right\} \right) \\ &\quad + |u_t|^{d/y_t} \log |u_t| \tilde{f}_{\text{sing}} \left( \frac{u_h}{|u_t|^{y_h/y_t}}, \left\{ \frac{u_j}{|u_t|^{y_j/y_t}} \right\} \right) . \end{aligned} \quad (22)$$



Table 2: *Extrapolation of  $a_{F_4}$* 

$\beta$	$-F^{(4)}t^{11/2}$	$\frac{-F^{(4)}u_t^{11/2}}{(u_h/h)^4}$	$+b_{F_4}t^{2.75}$	ext. $t^{3.75}$	ext. $t^4$
0.200	3.21111498292(2)	4.202506	4.358692		
0.250	3.4721394466(1)	4.286829	4.369154	4.383609	4.382846
0.280	3.6225400346(6)	4.321965	4.373381	4.381365	4.380954
0.300	3.720514859(3)	4.339751	4.375424	4.380395	4.380141
0.310	3.768883189(7)	4.347109	4.376235	4.380086	4.379911
0.320	3.81687386(2)	4.353517	4.376917	4.379774	4.379635
0.330	3.86451569(5)	4.359033	4.377479	4.379545	4.379438
0.340	3.91183946(6)	4.363716	4.377934	4.379382	4.379302
0.350	3.9588780(3)	4.367629	4.378292	4.379270	4.379213
0.355	3.9823015(7)	4.369315	4.378439	4.379239	4.379196
0.360	4.0056667(8)	4.370833	4.378566	4.379203	4.379168
0.365	4.028978(1)	4.372189	4.378674	4.379174	4.379146
0.370	4.052240(2)	4.373391	4.378765	4.379148	4.379126

Here  $F_b(t, h)$  is a regular function of  $t$  and  $h^2$ , the so-called bulk contribution,  $u_t$ ,  $u_h$ ,  $\{u_j\}$  are the non-linear scaling fields associated respectively to the temperature, the magnetic field and the irrelevant operators, and  $y_t$ ,  $y_h$ ,  $\{y_j\}$  are the corresponding dimensions. For the Ising model  $y_t = 1$ ,  $y_h = 15/8$ . Notice the presence of the logarithmic term, that is related to a “resonance” between the thermal and the identity operator<sup>1</sup>. The scaling fields are analytic functions of  $t$  and  $h$  that respect the  $\mathbf{Z}_2$  parity of  $t$  and  $h$ . Let us write the Taylor expansion for  $u_h$  and  $u_t$ , keeping only those terms that are needed for our analysis (we use the notations of [8]):

$$u_h = h [1 + c_h t + d_h t^2 + e_h h^2 + f_h t^3 + O(t^4, th^2)], \quad (23)$$

$$u_t = t + b_t h^2 + c_t t^2 + d_t t^3 + e_t t h^2 + g_t h^4 + f_t t^4 + O(t^5, t^2 h^2). \quad (24)$$

Let us first discuss the contributions of the irrelevant operators. In generic

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<sup>1</sup>In principle, logarithmic terms may also arise from additional resonances due to the fact that  $y_j$  are integers or differ by integers from  $y_h$ . They will not be considered here since these contributions either are subleading with respect to those we are interested in or have a form that is already included.

models their dimensions are usually unknown. In the present case instead, we may identify the irrelevant operators with the secondary fields obtained from the exact solution of the model at the critical point and use the corresponding RG exponents as input of our analysis. We shall discuss this issue in full detail in a forthcoming publication, let us only summarize here the main results of this analysis. It turns out that, discarding corrections of order  $O(t^5)$ , we have only two possible contributions:

- The first one is due to terms  $T\bar{T}$ ,  $T^2$  and  $\bar{T}^2$  (where  $T$  denotes the energy-momentum tensor). These terms would give a correction proportional to  $t^2$  in the scaling function.
- The second contribution is due to the  $L_{-3}\bar{L}_{-3}I$  field from the Identity family and to  $L_{-4}\epsilon$ ,  $\bar{L}_{-4}\epsilon$  from the energy family (where the  $L_{-i}$ 's are the generators of the Virasoro algebra). These terms give a correction proportional to  $t^4$  in the scaling function.

However, it turns out (see for instance the remarks of [2, 8, 10]) that in the infinite-volume free energy of the 2d Ising model the  $T\bar{T}$ ,  $T^2$  and  $\bar{T}^2$  terms are actually absent<sup>2</sup>. Thus, from the above analysis we see that the first correction due to the irrelevant fields appears only at order  $t^4$ . Therefore, since  $u_j/|u_t|^{y_j/y_t}$  vanishes for  $t \rightarrow 0$ , we can expand

$$f_{\text{sing}}(x, \{z_j\}) = Y_+(x) + u_0(t, h)u_t^4 X_+(x) + O(u_t^5), \quad (25)$$

where  $u_0(t, h)$  is an analytic function of  $t$  and  $h$ , and  $Y_+$ ,  $X_+$  are appropriate scaling functions. The same expansion holds for  $\tilde{f}_{\text{sing}}$  with different functions  $\tilde{Y}_+$ ,  $\tilde{X}_+$ . Additional constraints can be obtained using the exactly known results for the free energy, the magnetization and the susceptibility in zero field. Since all numerical data indicate that all zero-momentum correlation functions diverge as a power of  $t$  without logarithms for  $t \rightarrow 0$ , we assume as in Ref. [8] that  $\tilde{Y}_+(x)$  is constant, i.e.  $\tilde{Y}_+(x) = \tilde{Y}_0$ . The exact results for the free energy and the magnetization give then [7]

$$c_h = \frac{\beta_c}{\sqrt{2}}, \quad d_h = \frac{23\beta_c^2}{16}, \quad f_h = \frac{191\beta_c^3}{48\sqrt{2}}, \quad (26)$$

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<sup>2</sup>This conjecture is verified by the free energy and by the susceptibility at  $h = 0$  [10] and by the free energy  $F(0, h)$  [2]. Note that this is expected to be true only in the thermodynamic limit. In the finite-size scaling limit corrections that vanish like  $L_1^{-2}$  are indeed observed [11]. It is also not true for other observables, for instance, for the correlation length  $\xi$ .

$$c_t = \frac{\beta_c}{\sqrt{2}}, \quad d_t = \frac{7\beta_c^2}{6}, \quad f_t = \frac{17\beta_c^3}{6\sqrt{2}}, \quad (27)$$

where we have adapted the numbers of [7] to our normalizations, and  $\tilde{Y}_0 = -4\beta_c^2/\pi$ . By making use of the expansion of the susceptibility, we obtain further

$$Y_+^{(2)}(0) = A_\chi, \quad b_t = -\frac{D_0\pi}{16\beta_c^2}, \quad (28)$$

where  $D_0$  is the coefficient of the contribution proportional to  $t \log |t|$  in the susceptibility. Numerically  $D_0 = 0.04032550\dots$ , so that  $b_t = -0.0407708\dots$ . Nickel [10] has also conjectured, on the basis of the numerical analysis of the high-temperature series of the susceptibility, that  $e_t = b_t\beta_c\sqrt{2}$ .

Using the results presented above, and taking four derivatives of the free energy we obtain

$$\begin{aligned} F^{(4)} = & t^{-11/2} (a_{F4}(t) + t^4 \tilde{a}_{F4}(t) \log |t|) + t^{-11/4} (b_{F4}(t) + t^4 \tilde{b}_{F4}(t) \log |t|) \\ & + c_{F4}(t) + \tilde{c}_{F4}(t) \log |t|, \end{aligned} \quad (29)$$

where  $a_{F4}(t)$ ,  $b_{F4}(t)$ ,  $c_{F4}(t)$ ,  $\tilde{a}_{F4}(t)$ ,  $\tilde{b}_{F4}(t)$ , and  $\tilde{c}_{F4}(t)$  are analytic functions. Using Eqs. (26) and (27), we can compute the first terms in the Taylor expansion of  $a_{F4}(t)$ . By direct evaluation we find

$$\begin{aligned} a_{F4}(t) &= Y_+^{(4)}(0) \frac{(1 + c_h t + d_h t^2 + f_h t^3)^4}{(1 + c_t t + d_t t^2 + f_t t^3)^{11/2}} + O(t^4) \\ &= Y_+^{(4)}(0) \left( 1 - \frac{3\beta_c}{2\sqrt{2}} t + \frac{13\beta_c^2}{48} t^2 + \frac{29\beta_c^3}{32\sqrt{2}} t^3 \right) + O(t^4). \end{aligned} \quad (30)$$

From Eq. (30), we immediately identify

$$Y_+^{(4)}(0) = A_{F^{(4)}}. \quad (31)$$

Analogously, a direct calculation shows that

$$b_{F4}(0) = -21 b_t Y_+^{(2)}(0) = 0.8241504\dots \quad (32)$$

The contributions proportional to  $c_{F4}(t)$  and  $\tilde{c}_{F4}(t)$  give corrections of order  $t^{11/2}$  which will be neglected in the following.

Putting together the various terms, we end up with the following expression for the scaling function:

$$F^{(4)}t^{11/2} = A_{F^{(4)}}(1 + p_1t + p_2t^2 + p_3t^3) + p_4t^{11/4} + p_5t^{15/4} + p_6t^4 + \tilde{p}_6t^4 \log|t| + p_7t^{19/4} + O(t^5) \quad (33)$$

where

$$p_1 = -\frac{3\beta_c}{2\sqrt{2}} = -0.46741893... \quad (34)$$

$$p_2 = \frac{13\beta_c^2}{48} = 0.052597147... \quad (35)$$

$$p_3 = \frac{29\beta_c^3}{32\sqrt{2}} = 0.054843243... \quad (36)$$

$$p_4 = -21b_tY_+^{(2)}(0) = 0.8241504... \quad (37)$$

and  $p_5, p_6, \tilde{p}_6, p_7$  and  $A_{F^{(4)}}$  are undetermined constants which we shall try to fix in the next section.

## 5 Analysis of the data

The aim of this section is to obtain a numerical estimate for  $A_{F^{(4)}}$  by fitting the data reported in tab.2 with the scaling function (33). The major problem in doing this is to estimate the systematic errors involved in the truncation of the scaling function. To this end we performed two different types of analysis. Let us see in detail the procedure that we followed.

### 5.1 First level of analysis

We first performed a rather naive analysis of the data. In table 2 we include step by step the information that we have gained in the previous section. In the third column of table 2 we have multiplied  $-F^{(4)}$  by  $\frac{u_t^{11/2}}{(u_h/h)^4}$ , where  $u_h$  and  $u_t$  are given by eqs. (23,24). We see that the variation from  $\beta = 0.30$  to  $\beta = 0.37$  of the numbers in column three is reduced by a factor of about 10 compared with column two. In column four we add  $b_{F4}(0) t^{11/4}$  to the numbers of column three. Again we see that the variation of the numbers with  $\beta$  is drastically reduced in column four compared with column three.

Since we do not know the coefficients of higher order corrections exactly we have to extract them from the data. In the last two columns of table 2 we have extrapolated linearly in  $t^x$ , with  $x = 3.75$  in column 5 and  $x = 4$  in column 6. For the extrapolation we used neighboring  $\beta$ -values (e.g. the value quoted for  $\beta = 0.37$  is obtained from the extrapolation of the data for  $\beta = 0.365$  and  $\beta = 0.37$ ).

We see that the result of the extrapolation does not vary very much when the exponent is changed from  $15/4$  to  $4$ . Also the numbers given in column 5 and 6 are much more stable than those of column 4.

From this naive analysis we conclude that  $a_{F_4}(0) = 4.3791(1)$ , where the error bar is roughly estimated from an extrapolation of column 5 with  $t^4$ .

In the next section we shall try to include the higher order corrections in a more sophisticated fitting procedure.

## 5.2 Second level of analysis

We made three types of fits:

- f1] In the first we kept  $A_{F(4)}$ ,  $p_5$  and  $p_6$  as free parameters.
- f2] In the second we kept  $A_{F(4)}$ ,  $p_5$ ,  $p_6$  and  $p_7$  as free parameters.
- f3] In the third we kept  $A_{F(4)}$ ,  $p_5$ ,  $p_6$  and  $\tilde{p}_6$  as free parameters.

These are the only choices allowed by the data. If we neglect also  $p_6$  we can never obtain an acceptable confidence level (in fact we know that  $p_6$  is certainly different from zero and our data are too precise to allow such an approximation). If we add further terms, like a power of  $t^5$  for instance, or try to fit simultaneously  $p_5$ ,  $p_6$ ,  $\tilde{p}_6$  and  $p_7$  it always happens that some of the amplitudes are smaller than their statistical uncertainty signalling that our data are not precise enough to allow for five free parameters.

In order to estimate the systematic errors involved in the estimate of  $A_{F(4)}$  we performed for all the fitting functions several independent fits trying first to fit all the existing data (those listed in tab.2) and then eliminating the data one by one, starting from the farthest from the critical point. Among the set of estimates of the critical amplitudes we selected only those fulfilling the following requirements:

- 1] The reduced  $\chi^2$  of the fit must be of order unity. In order to fix precisely a threshold we required the fit to have a confidence level larger than 30%.
- 2] For all the subleading terms included in the fitting function, the amplitude estimated from the fit must be larger than the corresponding error, otherwise the term is eliminated from the fit. It is exactly this constraint which forbids us to take into account fits with more than four free parameters.
- 3] The amplitude of the  $n^{th}$  subleading field must be such that when it is multiplied for the corresponding power of  $t$ , (for the largest value of  $t$  involved in the fit) it gives a contribution smaller than that of the  $(n - 1)^{th}$  subleading term. This is intended to avoid artificial cancellations between subleading fields.

Among all the estimates of the critical amplitude  $A_{F(4)}$  fulfilling these requirements we select the smallest and the largest ones as lower and upper bounds.

The results of the fits are reported in tab.3, 4 and 5. We report all the combinations of input data which fulfill requirements [1]-[3]. In the tables we also report the best fit value of  $p_5$ . All the fits were performed using the double-precision NAG routine GO2DAF.

$A_{F(4)}$	$p_5$	d.o.f.
-4.3791092(9)	0.236(3)	4
-4.3791065(9)	0.220(12)	3
-4.379095(2)	0.142(18)	2

Table 3: *Fits of type [f1] fulfilling the requirements 1-3. In the first column the best fit results for the critical amplitude (with the error induced by the systematic errors of the input data in parenthesis), in the second column the best fit value of  $p_5$ , in the last column the number of degrees of freedom (i.e. the number of data used in the fit minus the number of free parameters).*

$A_{F^{(4)}}$	$p_5$	d.o.f.
$-4.3791003(1)$	$0.0486(7)$	7
$-4.3791001(3)$	$0.047(3)$	6
$-4.3791006(8)$	$0.053(9)$	5
$-4.3791022(14)$	$0.079(22)$	4

Table 4: *Same as tab.3, but with fits of type [f2].*

$A_{F^{(4)}}$	$p_5$	d.o.f.
$-4.3790944(1)$	$-0.692(1)$	8
$-4.3790942(3)$	$-0.696(7)$	7
$-4.3790961(6)$	$-0.634(17)$	6
$-4.3790980(10)$	$-0.56(4)$	5
$-4.3791005(16)$	$-0.41(8)$	4

Table 5: *Same as tab.3, but with fits of type [f3].*

Looking at the three tables and selecting the lowest and highest values of  $A_{F^{(4)}}$  we obtain the bounds

$$-4.379093 \gtrsim A_{F^{(4)}} \gtrsim -4.379110, \quad (38)$$

from which, using eq.(13), we obtain

$$g_4^* = 14.69735(3) \quad (39)$$

which we consider as our best estimate for  $g_4^*$ . As anticipated in the introduction, this result is in substantial agreement with the estimate of [1]. Notice however that the error quoted in eq.(39) should not be considered as a standard deviation. It rather encodes in a compact notation the systematic uncertainty of our fitting procedure.

We can compare the estimate (39) with previous numerical determinations. The analysis of high-temperature expansions gives  $g_4^* = 14.694(2)$ , Ref. [12] and  $g_4^* = 14.693(4)$ , Ref. [13] while Monte Carlo simulations give

$g_4^* = 14.3(1.0)$ , Ref. [14], and  $g_4^* = 14.69(2)$ , Ref. [1]. These results agree with our estimate (39), which is however much more precise.

It is clear from the data (see the second column of tab. 3, 4 and 5) that the uncertainty on  $A_{F(4)}$  is mostly due to the fluctuation of  $p_5$ . If one would be able to fix exactly also  $p_5$ , the precision in the determination of  $g_4$  could be significantly enhanced.

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